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Fracture mechanical assessment of interface cracks with contact zones in piezoelectric bimaterials under thermoelectromechanical loadings II. Electrically impermeable interface cracks

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Abstract

This paper constitutes the second part of a study of interface cracks with contact zones in thermopiezoelectrical bimaterials, and it is concerned with the case of an electrically impermeable interface crack. The principal physical peculiarity of this case in comparison with an impermeable interface crack is connected with the dependencies of the contact zone length and the fracture mechanical parameters on the prescribed electrical flux, and in a mathematical sense the main peculiarity is concerned with the reduction of the problem in question to the joint solution of inhomogeneous combined Dirichlet–Riemann and Hilbert boundary value problems. The exact analytical solutions of the mentioned problems have been found for an arbitrary contact zone length, and the required thermal, mechanical and electrical characteristics at the interface as well as the associated fracture mechanical parameters at the corresponding crack tips are presented. The transcendental equations for the determination of the real contact zone length have been obtained for a general case and for a small contact zone length in an especially simple form. Using the admissible directions of the heat and the electrical fluxes defined in this paper as well, the dependencies of the real contact zone length and the associated fracture and electrical intensity factors on the intensities of the thermal and electrical fluxes are presented in tables and associated diagrams.

Keywords: Electrically impermeable interface crack; Contact zone; Piezoelectric bimaterial; Heat flux

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1. Introduction

The investigation of an electrically permeable interface crack with a contact zone in a thermopiezoelectric bimaterial has been performed in Part 1 of this paper. It has been particularly shown that an electrical flux does not influence the fracture mechanical parameters in this case. Another limiting case of the electrical boundary conditions at the interface crack faces is based upon an assumption that the interface crack is completely impermeable for an electrical flux. In this case the physical situation at the crack region is principally different from the case of an electrically permeable interface crack that leads to the appearance of new essential peculiarities in the associated mathematical problem. Therefore, this type of electrical conditions at the interface crack surfaces values for a special consideration and it presents the subject of this part of the paper.

The applicability of simplified electric boundary conditions at the crack faces called the essential attention in the literature concerning the existence of cracks in a homogeneous piezoelectric material. In the references by Dunn and Taya (1993), Sosa and Khutoryansky (1996), Kogan et al. (1996), Zhang et al. (1998), Gao and Fan (1999) a slit crack has been considered as a limiting case of an elliptical hole or an inclusion and the exact electrical field in the mentioned hole or inclusion has been taken into account. By use of these solutions the authors arrived at the conclusion that the assumption of a permeable crack is generally more realistic than that of an impermeable crack, but in the absence of an electrical loading the J-integral values for an electrically impermeable slit crack are the same as for the case of a permeable crack. To the authors knowledge a similar investigation for an interface crack has not been presented in the literature yet, and therefore the validity of certain simplified electrical conditions at the crack faces has not been completely clarified concerning an interface crack. At present two simplified cases of the boundary conditions at the interface crack faces are actively used, i.e. the electrically impermeable crack and the electrically permeable crack, respectively, and in our opinion the first case has been more extensively used in the literature than the second one (Suo et al., 1992; Beom and Atluri, 1996; Shen and Kuang, 1998; Qin and Mai, 1999). Therefore, in this part of the paper the attention is focused upon an electrically impermeable interface crack mostly, regarding a comparison with an electrically permeable one studied in Part 1.

Concerning the investigation of an electrically impermeable interface crack in a thermomechanically loaded piezoelectric bimaterial the papers by Shen and Kuang (1998) and Qin and Mai (1999) should be mentioned. In the first of these papers the classical ("open" crack) model has been used while a thermally and electrically impermeable contact zone of an interface crack has been considered by Qin and Mai (1999) where the method of singular integral equations was applied. An electro-mechanically loaded electrically impermeable interface crack having a contact zone in a piezoelectric bimaterial has been analytically studied by Herrmann et al. (2001).

In the present study an electrically impermeable interface crack with a contact zone in a piezoelectric bimaterial under the action of a mechanical loading as well as thermal and electrical fluxes is considered. An exact analytical solution of the problem has been obtained and the particular cases related to the classical interface crack model and the contact zone models are considered.

2. Admissible values of heat and electrical fluxes

Ahead of the formulation and consideration of the main problem an auxiliary problem concerning a possible transition from a perfect thermo-electrical contact of two piezoelectric bodies to their thermal and electrical separation should be considered. For this purpose the same problem as in Section 5 of Part 1 of this paper should be taken into account where we assume now that the thermal insulator located in the region $|x_1| \leqslant a$ of the interface has the properties of the electrical insulation as well. Assuming that the part of the interface $|x_1| > a$ is mechanically frictionless, and in addition it is in a perfect thermal and electrical contact, the interface conditions for the perturbed thermal and the associated electromechanical problems can be presented in the form

$$\begin{aligned} &\text{for } |x_1| \leqslant a : q_3^{\pm} = q_0, \quad \sigma_{13}^{(m)}(x_1, 0) = 0, \quad \sigma_{33}^{(m)}(x_1, 0) = 0, \quad D_3^{(m)}(x_1, 0) = 0, \\ &\text{for } |x_1| > a : [T] = 0, \quad [q_3] = 0, \quad [u_3(x_1, 0)] = 0, \quad [\varphi(x_1, 0)] = 0, \end{aligned} \tag{1}$$

$$\sigma_{13}^{(m)}(x_1, 0) = 0, \quad [\sigma_{33}(x_1, 0)] = 0, \quad [D_3(x_1, 0)] = 0. \tag{2}$$

The designations of this part of the paper mostly coincide with those of Part 1. It is assumed by means of the conditions (1), (2) that in spite of the compressive load $\sigma_{33}^{(m)} = \sigma$ ($\sigma < 0$) at infinity the thermal and the electrical fluxes initiate an opening of the zone $|x_1| \leqslant a$ of a thermoelectrical insulation, i.e. they produce the appearance of an interface crack in this zone. The possibility of such a formulation is discussed in this section.

By using the method developed in Section 5 of Part 1 the presentations (44) of Part 1 can be found

$$\mathbf{S}(x_1) = \boldsymbol{\Psi}^{+}(x_1) - \boldsymbol{\Psi}^{-}(x_1), \tag{3a}$$

$$\mathbf{P}(x_1) = \mathbf{Q}\boldsymbol{\Psi}^{+}(x_1) - \overline{\mathbf{Q}}\boldsymbol{\Psi}^{-}(x_1) - \mathbf{e}\theta'^{+}(x_1) + \bar{\mathbf{e}}\theta'^{-}(x_1), \tag{3b}$$

where $\mathbf{S} = [\sigma_{13}^{(1)}, [u_3'], [\varphi']]^T$, $\mathbf{P} = [[u_1'], \sigma_{33}^{(1)}, D_3^{(1)}]^T$ and $\boldsymbol{\Psi}(z)$, $\mathbf{Q}$ and $\mathbf{e}$ are the same vectors as in Part 1 and

$$\theta'(z) = \frac{\mathrm{i}q_0}{2k_0}\sqrt{z^2 - a^2}. \tag{4}$$

It follows from the relation (3a) and the Eq. $\sigma_{13}^{(1)}(x_1, 0) = 0$ for $|x_1| \leqslant a$ that the function $\Psi_1(z)$ is analytic in the whole plane and because of the conditions at infinity $\Psi_1(z) \equiv C_1$ holds true. The value C_1 is an arbitrary constant which can be chosen to be zero due to an appropriate prescription of the stresses $\sigma_{11}^{(m)} = \sigma_{xm}^{\infty}$ at infinity.

Introducing the following matrix and vectors:

$$\boldsymbol{Q}_0 = \begin{bmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{bmatrix}, \quad \boldsymbol{e}_0 = \begin{Bmatrix} e_2 \\ e_3 \end{Bmatrix}, \quad \boldsymbol{\Psi}_0(z) = \begin{Bmatrix} \Psi_2(z) \\ \Psi_3(z) \end{Bmatrix}, \tag{5}$$

the boundary conditions (1_2), (1_3) lead to the following matrix Hilbert problem:

$$\boldsymbol{\Psi}_0^{+}(x_1) + \boldsymbol{\Psi}_0^{-}(x_1) = \boldsymbol{Q}_0^{-1}\boldsymbol{e}_0[\theta'^{+}(x_1) - \theta'^{-}(x)] \quad \text{for } |x_1| \leqslant a \tag{6}$$

with the following conditions at infinity:

$$\boldsymbol{\Psi}_0(z)|_{z\to\infty} = 0.5\boldsymbol{Q}_0^{-1}\begin{Bmatrix} \sigma \\ d \end{Bmatrix}. \tag{7}$$

By use of the formula (4) Eq. (6) can be rewritten in the form

$$\boldsymbol{\Psi}_0^{+}(x_1) + \boldsymbol{\Psi}_0^{-}(x_1) = \frac{\mathrm{i}q_0}{k_0}\boldsymbol{Q}_0^{-1}\boldsymbol{e}_0\sqrt{x_1^2 - a^2} \quad \text{for } |x_1| \leqslant a \tag{8}$$

and the solution of this equation can be presented according to Muskhelishvili (1977) as follows:

$$\boldsymbol{\Psi}_0(z) = -\frac{\mathrm{i}q_0}{k_0}\boldsymbol{Q}_0^{-1}\boldsymbol{e}_0\frac{1}{2\pi\mathrm{i}\sqrt{z^2 - a^2}}\int_{-a}^{a}\frac{t^2 - a^2}{t - z}\,\mathrm{d}t + \frac{\boldsymbol{a}_0 + \boldsymbol{a}_1 z}{\sqrt{z^2 - a^2}}, \tag{9}$$

where $\boldsymbol{a_i} = \begin{Bmatrix} a_{i1} \\ a_{i2} \end{Bmatrix}$ are arbitrary vectorial coefficients. Evaluating the integral in (9) and satisfying the condition at infinity (7) leads to the following solution of the problem (7), (8)

$$\boldsymbol{\Psi}_0(z)=\frac{1}{2}\boldsymbol{Q}_0^{-1}\left[\frac{q_0\boldsymbol{e}_0}{\pi k_0}\left(\frac{2az}{\sqrt{z^2-a^2}}+\sqrt{z^2-a^2}\ln\frac{z-a}{z+a}\right)+\left\{\begin{matrix}\sigma\\ d\end{matrix}\right\}\frac{z}{\sqrt{z^2-a^2}}\right]. \tag{10}$$

By use of the formulas (3) the expressions for the stress and the electrical displacement as well as for the derivatives of the mechanical displacement and electrical potential jumps can be written as follows:

$$\left\{\begin{matrix}\sigma_{33}^{(1)}(x_1,0)\\ D_3^{(1)}(x_1,0)\end{matrix}\right\}=\frac{q_0\boldsymbol{e}_0}{\pi k_0}\left(\frac{2ax_1}{\sqrt{x_1^2-a^2}}+\sqrt{x_1^2-a^2}\ln\frac{x_1-a}{x_1+a}\right)+\left\{\begin{matrix}\sigma\\ d\end{matrix}\right\}\frac{x_1}{\sqrt{x_1^2-a^2}}\quad\text{for } x_1>a, \tag{11a}$$

$$\left\{\begin{matrix}[u_3'(x_1)]\\ [\varphi'(x_1)]\end{matrix}\right\}=\mathrm{i}\boldsymbol{Q}_0^{-1}\left[\frac{q_0\boldsymbol{e}_0}{\pi k_0}\left(-\frac{2ax_1}{\sqrt{a^2-x_1^2}}+\sqrt{a^2-x_1^2}\ln\frac{a-x_1}{a+x_1}\right)-\left\{\begin{matrix}\sigma\\ d\end{matrix}\right\}\frac{x_1}{\sqrt{a^2-x_1^2}}\right]\quad\text{for } |x_1|<a. \tag{11b}$$

The formulas (11) are valid for any values of σ, q_0 and d, respectively, and it can be clearly seen that all functions defined by these formulas possess a square root singularity at the points $x_1=\pm a$. However, as it will be shown later the most important situation concerning the investigation of the contact zone model for a thermo- and an electro-impermeable interface crack is related to the case when the stress $\sigma_{33}^{(1)}(x_1,0)$ is non-singular at the points $x_1=\pm a$. It follows from the formula (11a) that such a situation can take place if

$$\sigma=-\frac{2ae_2q_0}{\pi k_0} \tag{12}$$

and the expressions for $\sigma_{33}^{(1)}(x_1,0)$ and $[u_3'(x_1)]$ attain the following form:

$$\sigma_{33}^{(1)}(x_1,0)=\frac{e_2q_0}{\pi k_0}\sqrt{x_1^2-a^2}\ln\frac{x_1-a}{x_1+a}\quad\text{for } x_1>a, \tag{13}$$

$$[u_3'(x_1)]=\Omega_{12}\left(\frac{2aq_0}{\pi k_0}e_3+d\right)\frac{x_1}{\sqrt{a^2-x_1^2}}-\frac{q_0}{\pi k_0}(\Omega_{11}e_2+\Omega_{12}e_3)\sqrt{a^2-x_1^2}\ln\frac{a-x_1}{a+x_1}\quad\text{for } |x_1|<a, \tag{14}$$

where $\boldsymbol{\Omega}=-\mathrm{i}\boldsymbol{Q}_0^{-1}$. In the formula (13) there is the single term connected with the logarithm, but in the formula (14) the leading term is connected with the square root singularity. Therefore, $\sigma_{33}^{(1)}(x_1,0)$ and $[u_3'(x_1)]$ are negative in the vicinity of the point $x_1=a$ (whereas $[u_3(x_1)]$ is positive) if the following inequalities are valid:

$$e_2q_0\geqslant 0,\quad \Omega_{12}\left(\frac{2aq_0}{\pi k_0}e_3+d\right)\leqslant 0. \tag{15}$$

It is worth to note that the inequality (15_1) and Eq. (12) agree with the associated relations for an electrically permeable crack presented in Part 1. However, in (15_1) we admit the possibility $e_2q_0=0$ because in this case $q_0=0$, $\sigma=0$ hold true, but d can differ from 0 and the bimaterial remains electrically loaded. The relations (12) and (15) define the values of q_0, σ and d for which a transition from a perfect thermoelectrical contact of two piezoelectric bodies to their thermal and electrical separation is possible. In spite of a particular problem dealt with, nevertheless the inequality (15) can be considered as a general condition because the relations (13), (14) define the asymptotic behaviour of $\sigma_{33}^{(1)}(x_1,0)$ and $[u_3'(x_1)]$, respectively, at the transition point $x_1=a$.

In a particular case when the left side of the inequality (15_2) becomes equal to zero it follows:

$$d=-\frac{2ae_3q_0}{\pi k_0} \tag{16}$$

and the second term of the formula (14) becomes the leading one. In this case for negative values of $\sigma_{33}^{(1)}(x_1,0)$ and $[u_3'(x_1)]$ in the vicinity of the point $x_1=a$ the following inequalities should be satisfied:

$$e_2 q_0 > 0, \quad q_0(\Omega_{11}e_2 + \Omega_{12}e_3) < 0. \tag{17}$$

The numerical analysis showed that for the considered bimaterials the signs of e_2 and $(\Omega_{11}e_2 + \Omega_{12}e_3)$ appear to be opposite. Therefore, in this particular case even the direction of the heat flux is defined by the inequality $(17)_1$ which agrees with the associated conclusion for an electrically permeable crack.

Introducing the following stress intensity factors:

$$k_1 = \lim_{x_1 \to a+0} \sqrt{2\pi(x_1 - a)}\sigma_{22}^{(1)}(x_1, 0), \quad k_4 = \lim_{x_1 \to a+0} \sqrt{2\pi(x_1 - a)}D_3^{(1)}(x_1, 0) \tag{18}$$

and using the formulas (11) one obtains

$$k_1 = 2a\sqrt{\pi a}q_0 e_2/(\pi k_0), \quad k_1^{(\mathrm{em})} = \sigma\sqrt{\pi a}, \tag{19a}$$

$$k_4 = 2a\sqrt{\pi a}q_0 e_3/(\pi k_0), \quad k_4^{(\mathrm{em})} = d\sqrt{\pi a}, \tag{19b}$$

where the superscripts (em) indicate the SIFs related to a pure electromechanical loading. By use of Eqs. (11) the asymptotic formulas for $\sigma_{33}^{(1)}(x_1, 0)$ and $[u_3'(x_1)]$ in the vicinity of the point $x_1 = a$ can be presented in the form

$$\sigma_{33}^{(1)}(x_1, 0)|_{x_1 \to a+0} = \frac{k_1 + k_1^{(\mathrm{em})}}{\sqrt{2\pi(x_1 - a)}} + \frac{k_1}{a}\sqrt{\frac{x_1 - a}{2\pi}}\ln\frac{x_1 - a}{2a}, \tag{20}$$

$$[u_3'(x_1)]|_{x_1 \to a-0} = [\Omega_{11}(k_1 + k_1^{(\mathrm{em})}) + \Omega_{12}(k_4 + k_4^{(\mathrm{em})})]\frac{1}{\sqrt{2\pi(a - x_1)}} - (\Omega_{11}k_1 + \Omega_{12}k_4)\frac{1}{a}\sqrt{\frac{a - x_1}{2\pi}}\ln\frac{a - x_1}{2a}. \tag{21}$$

Applying to the formulas (20) and (21) the same analysis as it was done concerning the formulas (11a) and (11b) Eqs. (12), (16) and the inequalities (15), (17) can easily be written in terms of the SIFs k_1, k_4 and $k_1^{(\mathrm{em})}$, $k_4^{(\mathrm{em})}$.

It is worth to note that the matrix $\boldsymbol{Q}_0$ and the vector $\boldsymbol{e}$ needed for applying the obtained formulas can be presented in terms of the matrix $\mathbf{G}$ and the vector $\mathbf{h}$ as follows:

$$\boldsymbol{Q}_0 = \begin{bmatrix} G_{33} - G_{31}G_{13}/G_{11} & G_{34} - G_{31}G_{14}/G_{11} \\ G_{43} - G_{41}G_{13}/G_{11} & G_{44} - G_{41}G_{14}/G_{11} \end{bmatrix}, \quad \boldsymbol{e} = \begin{Bmatrix} -h_1/G_{11} \\ -h_1G_{31}/G_{11} + h_3 \\ -h_1G_{41}/G_{11} + h_4 \end{Bmatrix}. \tag{22}$$

3. Formulation of the problem and the derivation of the basic relations

The statement of the problem and the initial stage of the solution are rather similar to the case of an electrically permeable crack and therefore many details in this part will be omitted for conciseness reasons.

Consider an interface crack situated in the region $c \leqslant x_1 \leqslant b$, $x_3 = 0$ between two different piezoelectric semi-infinite spaces $x_3 > 0$ and $x_3 < 0$ with thermomechanical properties defined by the matrices $E_{iJKl}^{(1)}$, $\lambda_{ij}^{(1)}$, $\beta_{iJ}^{(1)}$ and $E_{iJKl}^{(2)}$, $\lambda_{ij}^{(2)}$, $\beta_{iJ}^{(2)}$, respectively. The half-spaces are loaded at infinity with uniform stresses $\sigma_{33}^{(m)} = \sigma$, $\sigma_{13}^{(m)} = \tau$ and $\sigma_{11}^{(m)} = \sigma_{xm}^{\infty}$ as well as with uniform electric displacements $D_3^{(m)} = d$, $D_1^{(m)} = D_{xm}^{\infty}$ which satisfy the continuity conditions at the interface. Besides, a uniform temperature flux q_0 in the x_3-direction is imposed at infinity. As in the Part 1 of this paper it is assumed that the crack surfaces are traction-free for $x_1 \in [c, a] = L_1$ whilst they are in frictionless contact for $x_1 \in (a, b) = L_2$, and the position of the point $x_1 = a$ is arbitrarily chosen for the time being. The open part of the crack is thermally and electrically impermeable, whereas ideal thermoelectric contact takes place on the remaining part of the interface, see Fig. 2 in Part I. The interface conditions associated with such a formulation for the perturbed thermal state can be written as follows:

$$[T] = 0, \quad [q_3] = 0 \quad \text{for } x_1 \in (-\infty, \infty) \setminus (c, a), \tag{23a}$$

$$q_3^{\pm} = -q_0 \quad \text{for } x_1 \in L_1, \tag{23b}$$

$$[\mathbf{V}(x_1, 0)] = 0, \quad [\mathbf{t}(x_1, 0)] = 0 \quad \text{for } x_1 \in L, \tag{24a}$$

$$[\boldsymbol{t}^{(1)}(x_1, 0] = 0 \quad \text{for } x_1 \in L_1, \tag{24b}$$

$$[u_3(x_1, 0)] = 0, \quad [\varphi(x_1, 0)] = 0, \quad \sigma_{13}^{(m)}(x_1, 0) = 0, \quad [\sigma_{33}(x_1, 0)] = 0,$$

$$[D_3(x_1, 0)] = 0 \quad \text{for } x_1 \in L_2. \tag{24c}$$

For the considered problem the thermal solution is the same as in Part 1 and it can be presented in the form

$$\theta'(z) = \frac{\mathrm{i}q_0}{2k_0}\left[\sqrt{(z-c)(z-a)} - \tilde{z}\right], \tag{25}$$

and for the thermoelectromechanical solution the presentations (22)–(26) of Part 1 are valid. The most important part of this presentation can be rewritten as follows:

$$[\mathbf{V}'(x_1)] = \mathbf{W}^+(x_1) - \mathbf{W}^-(x_1), \tag{26}$$

$$\boldsymbol{t}^{(1)}(x_1, 0) = \boldsymbol{G}\boldsymbol{W}^+(x_1) - \overline{\boldsymbol{G}}\boldsymbol{W}^-(x_1) - \boldsymbol{g}(x_1), \tag{27}$$

where $\mathbf{V} = [u_1, u_2, u_3, \varphi]^T$, $\boldsymbol{t} = [\sigma_{31}, \sigma_{32}, \sigma_{33}, D_3]^T$, and $\mathbf{W}(z) = [W_1(z), W_2(z), W_3(z), W_4(z)]$ is a vector-function analytic in the whole plane with a cut along the crack region (c, b) with $\mathbf{g}(x_1) = \mathbf{h}\theta'^{+}(x_1) - \bar{\mathbf{h}}\theta'^{-}(x_1)$. Furthermore, thermopiezoelectric materials of the symmetry class 6 mm (Parton and Kudryavtsev, 1988) poled in the direction x_3 will be considered, and the attention will be focused upon the plane strain problem with the bimaterial matrix $\mathbf{G}$ and the vector $\mathbf{h}$ presented in the form (27) of Part 1.

Further, the transformation of the Eqs. (26), (27) which for a case of electromechanical loading are written in details in the paper by Herrmann et al. (2001) will be performed. Introducing a one row matrix $\mathbf{S} = [S_1, S_3, S_4]$ and considering a product $\mathbf{St}^{(1)}(x_1, 0)$ the following relations can be obtained using Eqs. (26), (27)

$$\sigma_{33}^{(1)}(x_1, 0) + m_{j4}D_3^{(1)}(x_1, 0) + \mathrm{i}m_{j1}\sigma_{13}^{(1)}(x_1, 0) = F_j^+(x_1) + \gamma_j F_j^-(x_1) - g_{0j}(x_1), \tag{28}$$

$$n_{j1}[u_1'(x_1)] + \mathrm{i}n_{j3}[u_3'(x_1)] + \mathrm{i}n_{j4}[\varphi'(x_1)] = F_j^+(x_1) - F_j^-(x_1), \tag{29}$$

where

$$F_j(z) = n_{j1}W_1(z) + \mathrm{i}[n_{j3}W_3(z) + n_{j4}W_4(z)], \tag{30}$$

$g_{0j}(x_1) = g_3(x_1) + im_{j1}g_1(x_1) + m_{j4}g_4(x_1)$, $m_{j4} = S_{j4}$, $m_{j1} = -\mathrm{i}S_{j1}$, $n_{j1} = Y_{j1}$, $n_{j3} = -\mathrm{i}Y_{j3}$, $n_{j4} = -\mathrm{i}Y_{j4}$ and m_{jl}, n_{jl} $(l = 1, 3, 4)$ are real. Moreover $\boldsymbol{Y}_j = \boldsymbol{S}_j\boldsymbol{G}$ and γ_j, $\boldsymbol{S}_j^T = [S_{j1}, S_{j3}, S_{j4}]$ $(j = 1, 3, 4)$ are the eigenvalues and eigenvectors of the matrix $(\gamma\boldsymbol{G}^T + \overline{\boldsymbol{G}}^T)$, respectively. The roots of the equation $\det(\gamma\boldsymbol{G}^T + \overline{\boldsymbol{G}}^T) = 0$ can be presented in the form

$$\gamma_1 = \frac{1+\delta}{1-\delta}, \quad \gamma_3 = \gamma_1^{-1}, \quad \gamma_4 = 1, \tag{31}$$

where

$$\delta^2 = \frac{g_{14}^2 g_{33} + g_{13}^2 g_{44} - 2g_{14}g_{13}g_{34}}{g_{11}(g_{33}g_{44} - g_{34}^2)}. \tag{32}$$

The numerical analysis shows that for one group of piezoelectric bimaterials the inequality

$$\delta^2 > 0 \tag{33}$$

holds true, while for another group this inequality is not valid. Thus, attention is paid in the following to those bimaterials satisfying the inequality (33), and the properties of the coefficients m_{jl}, n_{jl} reported above are related to these materials.

Because of the existing linearity the formulated problem can be considered separately for an electromechanical and a thermal loading, respectively. Taking into account that the problem in question under a pure electromechanical loading has been already studied in detail by Herrmann et al. (2001), thermal loading only will be further considered assuming $\sigma = \tau = \sigma_{xxm}^{\infty} = 0$ and $d = D_{xm}^{\infty} = 0$ for the time being. Moreover, taking into account that all fields called by the perturbed thermal state disappear for large z, the conditions at infinity for the functions $F_j(z)$ by using of Eq. (28) can be written as follows:

$$F_j(z) = 0 \quad \text{for } z \to \infty. \tag{34}$$

The obtained relations (28), (29) and (34) by considering the properties of the coefficients m_{jl} and n_{jl} are rather convenient for the formulation and the analysis of the problems of linear relationship for an electrically impermeable interface crack with a contact zone.

4. Formulation and solution of the problems of linear relationship

Satisfying the boundary conditions (24b) by means of Eq. (28) gives

$$F_j^+(x_1) + \gamma_j F_j^-(x_1) = g_{0j}(x_1) \quad \text{for } x_1 \in L_1 \quad (j = 1, 3, 4). \tag{35}$$

Using the fact that $m_{41} = 0$, $n_{41} = 0$ and satisfying the first three of the boundary conditions (24c) by means of Eqs. (28), (29) leads to the following relations:

$$\mathrm{Im} F_k^{\pm}(x_1) = \frac{1}{1 + \gamma_k} \mathrm{Im}\{g_{0k}(x_1)\} \quad \text{for } x_1 \in L_2 \quad (k = 1, 3), \tag{36a}$$

$$F_4^+(x_1) - F_4^-(x_1) = 0. \tag{36b}$$

Taking into account that the problem (35), (36a) for $j = k = 3$ can be obtained from the same problem for $j = k = 1$, in future this problem will be considered for $j = 1$ and $k = 1$ only. Using formula (25) leads to the following form of Eqs. (35), (36a) for $j = k = 1$

$$F_1^+(x_1) + \gamma_1 F_1^-(x_1) = \frac{q_0}{k_0} \varphi_1(x_1) \quad \text{for } x_1 \in (c, a), \tag{37}$$

$$\mathrm{Im} F_1^{\pm}(x_1) = \frac{q_0}{k_0} \varphi_2(x_1) \quad \text{for } x_1 \in (a, b), \tag{38}$$

where

$$\varphi_1(x_1) = \mathrm{i} m_{11} \theta_1 \tilde{x}_1 + \mathrm{i}(\theta_3 + m_{14}\theta_4)\sqrt{(x_1 - c)(x_1 - a)}, \tag{39}$$

$$\varphi_2(x_1) = \frac{m_{11}\theta_1}{1 + \gamma_1}\left[\tilde{x}_1 - \sqrt{(x_1 - c)(x_1 - a)}\right] \tag{40}$$

and $\tilde{x}_1 = x_1 - c_0$, $c_0 = (c + a)/2$. It is worth to remind as well that the values θ_1, θ_3, θ_4 define the vector $\mathbf{h}$ which has been introduced by the formula (27) of Part 1. The relations (37) and (38) represent an inhomogeneous combined Dirichlet–Riemann boundary value problem for the sectionally holomorphic function $F_1(z)$ which was considered in detail in Part 1. The Eq. (34) for $j = 1$ can be used as a condition at infinity for this problem.

Using Eq. (35) for $j = 4$ as well as Eqs. (25) and (36b) one arrives at a Hilbert problem for the function $F_4(z)$ which is analytical in the whole plane with a cut along L_1 only

$$F_4^+(x_1) + F_4^-(x_1) = \frac{q_0}{k_0}\varphi_3(x_1) \quad \text{for } x_1 \in (c, a), \tag{41a}$$

where

$$\varphi_3(x_1) = \mathrm{i}(\theta_3 + m_{44}\theta_4)\sqrt{(x_1 - c)(x_1 - a)}. \tag{41b}$$

Eq. (34) for $j = 4$ can be used as a condition at infinity for this problem.

The inhomogeneous combined Dirichlet–Riemann problem is principally the same as the associated problem of Part 1. The differences are due to the coefficients of the functions (39), (40) and the meaning of the function $F_1(z)$. Therefore, the solution of this problem will be presented here only. Thus, the function $F_1(z)$ has the following form:

$$F_1(z) = q_0k_0^{-1}RX_1(z) + q_0k_0^{-1}X_2(z)[\omega_1(z) + \omega_2(z)], \tag{42}$$

where

$$R = \frac{\eta_1 + \eta_3}{\pi}\int_a^b (t - c)\sqrt{\frac{t - a}{b - t}}\cosh\varphi_0(t)\,\mathrm{d}t + R_0, \tag{43a}$$

$$R_0 = \left[\frac{(b + a)^2}{4} + \frac{(b - a)^2}{8}\right]d_1 + \frac{b + a}{2}d_2 + d_3, \tag{43b}$$

$$\omega_1(z) = \frac{1}{2\pi \mathrm{i}}\int_c^a \frac{\varphi_1(t)\,\mathrm{d}t}{X_2^+(t)(t - z)}, \tag{44a}$$

$$\begin{aligned}\omega_2(z) = {} & \frac{\eta_1 + \eta_3}{\pi}Y(z)\int_a^b (t - c)\sqrt{\frac{t - a}{b - t}}\cosh\varphi_0(t)\,\mathrm{d}t + \mathrm{i}[d_1z^2 + d_2z + d_3 - Y(z)(d_1(z + c_0) + d_2)] \\ & + \frac{\eta_1}{\pi}\int_a^b \frac{\tilde{t}\sqrt{(t - c)(t - a)} - (t - c)(t - a)}{t - z}\sinh\varphi_0(t)\,\mathrm{d}t\end{aligned} \tag{44b}$$

with $Y(z) = \sqrt{(z - a)(z - b)}$, $d_1 = -(\eta_1 + \eta_3)\cos\beta$, $d_2 = (\eta_1 + \eta_3)[\beta_1\sin\beta + (a + c)\cos\beta]$,

$$d_3 = (\eta_1 + \eta_3)\left\{\left(\frac{\beta_1^2}{2} - ac\right)\cos\beta + \beta_1\frac{b - 3a - 2c}{4}\sin\beta\right\} - \eta_3\frac{(a - c)^2}{8}\cos\beta, \tag{45}$$

where $\eta_1 = -m_{11}\theta_1/(\gamma_1 + 1)$, $\eta_3 = \theta_3 + m_{14}\theta_4/(\gamma_1 - 1)$. All remaining values used in the formulas (42)–(45) are the same as in Part 1.

The Hilbert problem (41) is principally very similar to the problem (7), (8) considered above. Therefore, dropping the details of the solution the function $F_4(z)$ satisfying the required condition at infinity reads as following:

$$F_4(z) = \frac{q_0}{2\pi k_0}(\theta_3 + m_{44}\theta_4)\left[\frac{(a-c)\tilde{z}}{\sqrt{(z-c)(z-a)}} + \sqrt{(z-c)(z-a)}\ln\frac{z-a}{z-c}\right]. \tag{46}$$

Using the solutions (42), (46) and formula (28) the normal stress and the electrical displacement at the interface can be found from the following system:

$$\sigma_{33}^{(1)}(x_1,0) + m_{14}D_3^{(1)}(x_1,0) = \mathrm{Re}\{F_1^+(x_1) + \gamma_1 F_1^-(x_1) - q_0k_0^{-1}\varphi_1(x_1)\}, \tag{47a}$$

$$\sigma_{33}^{(1)}(x_1,0) + m_{44}D_3^{(1)}(x_1,0) = F_4^+(x_1) + F_4^-(x_1) - q_0k_0^{-1}\varphi_3(x_1) \tag{47b}$$

and the shear stress is defined by the formula

$$\sigma_{13}^{(1)}(x_1,0) = m_{11}^{-1}\mathrm{Im}\{F_1^+(x_1) + \gamma_1 F_1^-(x_1) - q_0k_0^{-1}\varphi_1(x_1)\}. \tag{48}$$

The derivatives of the normal displacement and the electrical potential jumps at the interface can be found in a similar manner by means of the formulas (29), (42) and (46) from the following system:

$$n_{13}[u_3'(x_1)] + n_{14}[\varphi'(x_1)] = \mathrm{Im}\{F_1^+(x_1) - F_1^-(x_1)\}, \tag{49a}$$

$$n_{43}[u_3'(x_1)] + n_{44}[\varphi'(x_1)] = F_4^+(x_1) - F_4^-(x_1) \tag{49b}$$

and the derivative of the transversal displacement jump reads as follows:

$$[u_1'(x_1)] = n_{11}^{-1}\{F_1^+(x_1) - F_1^-(x_1)\}. \tag{50}$$

By means of the Eqs. (47), (50) all required components of the stress-strain state can be found at any point of the material interface.

Nevertheless, in spite of the fact that the obtained solution is derived on an exact analytical way there are still remaining integrals in the formulas (43a) and (44). These integrals can be evaluated numerically, but for a very small relative contact zone length $\lambda = (b-a)/(b-c)$ such an evaluation can be connected with some difficulties. Fortunately for a small λ the solution (42)–(44) can be essentially simplified and an asymptotic expression for $F_1(z)$ can be obtained. Presenting $\omega_1(z)$ (44a) in the form

$$\omega_1(z) = \frac{e^{-i\beta}}{2\pi i}\int_c^a \frac{\varphi_1(t)\sqrt{(t-c)(t-a)}}{t-x} e^{i[\beta-\varphi(t)]}\,dt \tag{51}$$

and taking into account that for a small λ an approximation $\beta - \varphi(t) \approx \varepsilon\ln\left(\frac{t-a}{t-c}\right)$ is valid (Herrmann and Loboda, 2001) the integral (51) can be approximately evaluated and presented in the form

$$\omega_1(z) \approx \tilde{\omega}_1(z)$$
$$= ie^{-i\beta}\eta_1\left\{-\frac{\tilde{z}}{X(z)} + [d_{11}z^2 + d_{12}z + d_{13}]\right\} + ie^{-i\beta}\eta_3\left\{-\frac{\sqrt{(z-c)(z-a)}}{X(z)} + [d_{21}z^2 + d_{22}z + d_{23}]\right\}, \tag{52}$$

where $X(z)$ and d_{ij} are the same as in the formulas (57), (58) of Part 1. Further by neglecting the integrals in the formulas (43a) and (44b) the following formula for $\widetilde{F}_1(z) \approx F_1(z)$ can be obtained

$$\widetilde{F}_1(z) = q_0k_0^{-1}R_0X_1(z) + q_0k_0^{-1}X_2(z)[\tilde{\omega}_1(z) + \tilde{\omega}_2(z)], \tag{53a}$$

where

$$\tilde{\omega}_2(z) \approx \omega_2(z) = i[d_1z^2 + d_2z + d_3 - Y(z)(d_1(z+c_0) + d_2)]. \tag{53b}$$

The stresses and the electrical displacement as well as the derivatives of the normal displacement and the electrical potential jumps at the interface can be found from the Eqs. (47)–(50) in which $\widetilde{F}_1^{\pm}(x_1)$ instead of

$F_1^{\pm}(x_1)$ should be taken. All formulas in this case appear to be extremely simple and according to the numerical verification they can be used not only for very small but also for moderate values of λ.

5. Behavior of the solution at singular points

As can be seen from the results of the previous section the solution for an electrically impermeable crack essentially differs from a permeable one, first of all because of the appearance of new components connected with the function $F_4(z)$. Nevertheless, in this case also the normal stress $\sigma_{33}^{(1)}(x_1,0)$ remains limited for $x_1 \to b+0$ and the main stress and electrical displacement intensity factors (IFs) can be defined as follows:

$$\begin{aligned} k_1 &= \lim_{x_1 \to a+0} \sqrt{2\pi(x_1-a)}\sigma_{33}^{(1)}(x_1,0), \quad k_2 = \lim_{x_1 \to b+0} \sqrt{2\pi(x_1-b)}\sigma_{13}^{(1)}(x_1,0), \\ k_4 &= \lim_{x_1 \to a+0} \sqrt{2\pi(x_1-a)}D_3^{(1)}(x_1,0). \end{aligned} \tag{54}$$

Using the exact formulas (47), (48) leads to the following expressions for the IFs:

$$k_1 + m_{14}k_4 = \frac{q_0}{k_0}\sqrt{\frac{2}{\pi(a-c)}}I_0, \quad k_1 + m_{44}k_4 = \frac{q_0}{k_0}\sqrt{\frac{a-c}{2\pi}}(a-c)(\theta_3 + m_{44}\theta_4), \tag{55}$$

$$k_2 = -(1+\gamma_1)\frac{q_0}{m_{11}k_0}\sqrt{\frac{2\pi}{l}}R, \tag{56}$$

where

$$\begin{aligned} I_0 = {} & \int_c^a \sqrt{\frac{t-c}{a-t}}\Big[-m_{11}\theta_1\tilde{t}\sin\varphi^*(t) + (\theta_3 + m_{14}\theta_4)\sqrt{(t-c)(a-t)}\cos\varphi^*(t)\Big]\mathrm{d}t \\ & - 2\eta_1\sqrt{\gamma_1}\int_a^b \left[(t-c) - \tilde{t}\sqrt{\frac{t-c}{t-a}}\right]\sinh\varphi_0(t)\,\mathrm{d}t. \end{aligned} \tag{57}$$

It is obvious that the IFs k_1 and k_4 can be found from the system (55), and the quantity I_0 depends on integrals which can be obtained by a numerical calculation.

An evaluation of the Eqs. (49), (50) for $x_1 \to a-0$ and $x_1 \to b-0$, respectively, leads to the following asymptotic expressions:

$$n_{13}[u_3'(x_1)] + n_{14}[\varphi'(x_1)]|_{x_1 \to a-0} = -\sqrt{\frac{\alpha}{\gamma_1}}\frac{k_1 + m_{14}k_4}{\sqrt{2\pi(a-x_1)}}, \tag{58a}$$

$$n_{43}[u_3'(x_1)] + n_{44}[\varphi'(x_1)]|_{x_1 \to a-0} = -\frac{k_1 + m_{44}k_4}{\sqrt{2\pi(a-x_1)}}, \tag{58b}$$

$$[u_1'(x_1)]|_{x_1 \to b-0} = -\frac{\Theta_{22}k_2}{\sqrt{2\pi(a-x_1)}}, \tag{59}$$

where $\alpha = (\gamma_1+1)^2/4\gamma_1$ and $\Theta_{22} = -2m_{11}/n_{11}(1+\gamma_1)$. The solution of the system (58) gives the following result:

$$[u_3'(x_1,0)]|_{x_1 \to a-0} = -\frac{1}{\sqrt{2\pi(a-x_1)}}(\Theta_{11}k_1 + \Theta_{14}k_4), \tag{60a}$$

$$[\varphi'(x_1,0)]|_{x_1\to a-0} = -\frac{1}{\sqrt{2\pi(a-x_1)}}(\Theta_{41}k_1+\Theta_{44}k_4), \tag{60b}$$

where

$$\begin{aligned}\Theta_{11} &= \left(n_{44}\sqrt{\alpha/\gamma_1}-n_{14}\right)/\Delta_n; \quad \Theta_{14} = \left(m_{14}n_{44}\sqrt{\alpha/\gamma_1}-m_{44}n_{14}\right)\Big/\Delta_n,\\ \Theta_{41} &= \left(n_{13}-n_{43}\sqrt{\alpha/\gamma_1}\right)/\Delta_n; \quad \Theta_{44} = \left(m_{44}n_{13}-m_{14}n_{43}\sqrt{\alpha/\gamma_1}\right)\Big/\Delta_n\end{aligned} \tag{60c}$$

and $\Delta_n = n_{13}n_{44} - n_{43}n_{14}$. It is worth to note that that the asymptotic expressions (59), (60) completely coincide with the associated expressions for a pure mechanical loading obtained by Herrmann et al. (2001). Therefore, the formulas for the energy release rates (ERRs) defined for the points $x_1 = a$ and $x_1 = b$, respectively, in the just mentioned paper will be the same as well and they read as follows:

$$G_1^c = [\Theta_{11}k_1^2 + \Theta_{44}k_4^2 + (\Theta_{14}+\Theta_{41})k_1k_4]/4, \tag{61a}$$

$$G_2^c = \Theta_{22}k_2^2/4. \tag{61b}$$

Therefore, by existence of a small λ the use of the functions $\widetilde{F}_1^{\pm}(x_1)$ instead of $F_1^{\pm}(x_1)$ in the formulas (47a), (48) gives the following approximate expressions for the IFs:

$$\sqrt{\alpha}(k_1+m_{14}k_4) - \mathrm{i}m_{11}k_2 \approx \sqrt{\alpha}(\tilde{k}_1+m_{14}\tilde{k}_4) - \mathrm{i}m_{11}\tilde{k}_2 = \frac{q_0}{k_0}\sqrt{\alpha}T_1(\lambda), \tag{62a}$$

where

$$T_1(\lambda) = -\mathrm{i}\frac{l\sqrt{\pi l\gamma_1}}{2\sqrt{2}}\mathrm{e}^{\mathrm{i}\beta}[(1+2\mathrm{i}\varepsilon)^2(\eta_1+\eta_3)-\eta_3]. \tag{62b}$$

The assumption of a small λ in Eq. $(55)_2$ and by setting $(a-c) = l$ leads by a combination of this equation with Eq. (62a) to the following expressions for the IFs:

$$\tilde{k}_1 = (m_{44}-m_{14})^{-1}\frac{q_0}{k_0}\{m_{44}\mathrm{Re}[T_1(\lambda)] - m_{14}T_2\}, \tag{63a}$$

$$\tilde{k}_2 = -\frac{q_0\sqrt{\alpha}}{k_0m_{11}}\mathrm{Im}[T_1(\lambda)], \tag{63b}$$

$$\tilde{k}_4 = (m_{44}-m_{14})^{-1}\frac{q_0}{k_0}\{-\mathrm{Re}[T_1(\lambda)] + T_2\}, \tag{63c}$$

where $T_2 = l^{3/2}(\theta_3 + m_{44}\theta_4)/\sqrt{2\pi}$. It is worth to note that for small values of λ the formulas (59), (60) defining the behavior of the derivatives of the displacements and the electrical potential jumps at the singular points of the material interface remain valid by using $\tilde{k}_l$ instead of k_l $(l = 1, 2, 4)$ in their right-hand sides.

For a pure electromechanical loading according to Herrmann et al. (2001) the IFs for a small λ are defined by the following relations:

$$\sqrt{\alpha}(\tilde{k}_1^{(\mathrm{em})} + m_{14}\tilde{k}_4^{(\mathrm{em})}) - \mathrm{i}m\tilde{k}_2^{(\mathrm{em})} = \sqrt{\frac{\pi l}{2}}\mathrm{e}^{\mathrm{i}\beta}(1+2\mathrm{i}\varepsilon)(\sigma + m_{14}d - \mathrm{i}m_{11}\tau), \tag{64a}$$

$$\tilde{k}_1^{(\mathrm{em})} + m_{14}\tilde{k}_4^{(\mathrm{em})} = \sqrt{\frac{\pi l}{2}}(\sigma + m_{44}d) \tag{64b}$$

and for a combination of electromechanical and thermal loading they must be found as a sum of the correspondent IFs (63), (64). For a general case of loading and the essential values of λ the IFs can be found from the formulas (55), (56) for a thermal loading and from the associated formulas of the paper by Herrmann et al. (2001) for an electromechanical loading. However, we pay less attention to this case because mostly the contact zone length is extremely small and the expressions (63), (64) appear to be the most important for the analysis of the most real situations.

6. Contact zone models

The solution of the interface crack problem obtained in the previous chapter is mathematically correct for any position of the point $x_1 = a$. But for an arbitrary value of a this solution is not always physically admissible, and therefore, the correspondent interface crack model has been called by Herrmann et al. (2001) an artificial contact zone model (ACZM). The necessary additional conditions required for the physical correctness of the ACZM coincide with the associated conditions of Part 1, and they read as follows:

$$\sigma_{33}^{(1)}(x_1, 0) \leqslant 0 \quad \text{for } x_1 \in L_2, \quad [u_3(x_1, 0)] \geqslant 0 \quad \text{for } x_1 \in L_1. \tag{65}$$

An analytical analysis and numerical verifications show that these inequalities hold true if a is taken from the segment $[a_1, a_2]$ providing $a_1 \leqslant a_2$, where

$$a = b - \lambda l, \quad a_1 = b - \lambda_1 l, \quad a_2 = b - \lambda_2 l \tag{66}$$

and λ_1 is the maximum root taken from the interval $(0, 1)$ of the equation

$$k_1 + k_1^{(\mathrm{em})} = 0 \tag{67}$$

and λ_2 is the similar root of the equation

$$\sqrt{a - x_1}\{[u_3'(x_1, 0)] + [u_3'(x_1, 0)]^{(\mathrm{em})}\} = 0. \tag{68}$$

For arbitrary values of λ_1 and λ_2 these equations can be analytically formulated by means of the formulas (55), (60a) and the associated formulas for a pure electromechanical loading (Herrmann et al., 2001), whereas for small values of λ_1 and λ_2 Eqs. (67) and (68) by use of (60a), (63a), (64a), (64b) can be rewritten into the following relations, respectively

$$\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\beta}(1 + 2\mathrm{i}\varepsilon)[1 + m_{14}d/\sigma - \mathrm{i}m_{11}k - p\mathrm{i}(m_3 + 2\mathrm{i}\varepsilon m_4)]\right\} = \sqrt{\alpha}m_{14}m_{44}^{-1}\chi_3, \tag{69a}$$

$$\mathrm{Re}\left\{e^{i\beta}(1 + 2i\varepsilon)[1 + m_{14}d/\sigma - im_{11}k - pi(m_3 + 2i\varepsilon m_4)]\right\} = \sqrt{\gamma_1}n_{14}n_{44}^{-1}\chi_3, \tag{69b}$$

where

$$k = \frac{\tau}{\sigma}, \quad m_3 = m_1 - \frac{4\varepsilon^2}{1 + 4\varepsilon^2}m_2, \quad m_4 = m_1 - \frac{2 + 4\varepsilon^2}{1 + 4\varepsilon^2}m_2,$$

$$m_1 = \frac{-m_{11}\theta_1}{\vartheta k_0 \delta_2(1 + \gamma_1)}, \quad m_2 = \frac{\theta_3 + m_{14}\theta_4}{\vartheta k_0 \delta_2(1 - \gamma)}, \quad \chi_3 = 1 + m_{44}\frac{d}{\sigma} + lq_0\frac{\theta_3 + m_{44}\theta_4}{\pi\sigma k_0}. \tag{70}$$

The non-dimensional parameter $p = ((\gamma_1 + 1)/4)q_0\vartheta\delta_2 l/\sigma$ has the same meaning as in Part 1, and the parameters T and δ_2 are also the same as in Part 1.

It can be seen that the left-hand sides of Eqs. (69) are very similar to the associated equation of the electrically permeable crack, but the right hand sides are not trivial as in the case of the electrically permeable crack, and moreover, they differ from each other. Recollecting that $\beta = \varepsilon \ln[(1-\sqrt{1-\lambda}) \times (1+\sqrt{1-\lambda})^{-1}]$ while the other components of Eqs. (69) are independent of λ, the exact analytical solutions of these equations can be presented in the following form:

$$\lambda_i \approx \tilde{\lambda}_i = 4\exp(g_i/\varepsilon), \tag{71a}$$

where

$$\begin{aligned} & g_i = (-1)^n\left[\sin^{-1}\left(\frac{\chi_1 + 2\varepsilon\chi_2}{\Delta}\right) - \sin^{-1}(\zeta_i)\right] + \pi n, \\ & \chi_1 = 1 + m_{14}\frac{d}{\sigma} + 2\varepsilon p m_4, \quad \chi_2 = k m_{11} + p m_3, \\ & \zeta_1 = \sqrt{\alpha}\frac{m_{14}}{\Delta m_{44}}\chi_3, \quad \zeta_2 = \sqrt{\gamma_1}\frac{n_{14}}{\Delta n_{44}}\chi_3, \quad \Delta^2 = (1+4\varepsilon^2)(\chi_1^2 + \chi_2^2), \quad i = 1,2 \end{aligned} \tag{71b}$$

and n should be taken to choose the maximum roots of the Eqs. (69) from the interval $(0, 1)$. The formulas (71) are appropriate for small values of λ_i (approximately for $\lambda_i < 0.01$). However, if the values of $\tilde{\lambda}_1$ or $\tilde{\lambda}_2$ are of essential magnitude then the numerical solutions of Eq. (67) or Eq. (68), respectively, should be found and the precise magnitudes of λ_1 or λ_2 have to be calculated.

The situation arising here is not traditionally for the determination of the real contact zone length. Usually the real position of the point $x_1 = a$ is uniquely defined by the inequalities (65), and the contact zone model (CZM) in the Comninou (1977) sense takes place. However, in the reported case a set of positions $a \in [a_1, a_2]$ provided $(a_1 \leqslant a_2)$ holds true satisfy the inequalities (65). In other terms this set can be defined as follows:

$$\Omega_a = [a \geqslant a_1 \cap a \leqslant a_2]. \tag{72}$$

The meaning of the expression (72) can be explained in the following way. For a certain bimaterial and an associated loading the first inequality (65) is valid for any position of the point $x_1 = a$ satisfying the inequality $a \geqslant a_1$ and the second inequality (65) is valid for any $a \leqslant a_2$. Therefore, the set Ω_a is formed as the intersection of the solutions of the mentioned inequalities.

The appearance of the set Ω_a instead of a unique solution which usually takes place is connected first of all with the action of the electrical flux d. Therefore, it is not surprising that the size of the set Ω_a essentially depends on the quantity d. As it will be demonstrated later for the thermoelectromechanical case, and it was already shown by Herrmann et al. (2001) for a pure electromechanical loading, the set Ω_a is usually not empty for $d = 0$, but a_1 and a_2 differ very slightly. The increase of d leads to a decrease of a_1 and an increase of a_2, respectively, i.e. their difference increases. On the other hand a decrease of d leads to a decrease of $(a_2 - a_1)$ till to the point when $a_2 = a_1$. In this case $\Omega_a = a_2 = a_1$ holds true, and the unique solution of (65) exists. Further decreasing of d gives $\Omega_a = \emptyset$, and the contact zone model defined by the boundary conditions (24) does not exist.

The most characteristic situation is connected with $\Omega_a \neq \emptyset$, and it is clear that for any of such cases a unique contact zone defined by a real position of the point $x_1 = a$ should exist. In the paper by Herrmann et al. (2001) a position in question has been defined by means of the theorem of the minimum potential energy (Sokolnikoff, 1956). Taking into account that for a thermal loading the signs of $\Theta_{i,j}$ $(i, j = 1, 4)$ remain the same as for a pure electromechanical loading the conclusion of Herrmann et al. (2001) remains valid, and in the considered case, and therefore *for a thermoelectromechanical loading a real position of the point $x_1 = a$ coincides with a_1 provided $\Omega_a \neq \emptyset$ holds true.*

7. The classical model

For the sake of completeness the main results for the classical interface crack model ($b = a$, $L_2 = \emptyset$) will be presented here. In this case a perturbed thermal state is defined by Eqs. (37), (41) with the condition at infinity (34) providing the absence of an electromechanical loading. The solution (46) of the problem (34), (41) remains valid here while the solution of the Hilbert problem (34), (37) obtained in the same way as in Section 6 of Part 1 can be written in the form

$$F_1(z) = \frac{i\theta_1 q_0}{k_0(1+\gamma_1)}\left\{\tilde{z} - X(z)[d_{11}z^2 + d_{12}z + d_{13}]\right\} + \frac{i(\theta_3 + m_{14}\theta_4)q_0}{k_0(1-\gamma_1)}\left\{\sqrt{(z-c)(z-a)} - X(z)[d_{21}z^2 + d_{22}z + d_{23}]\right\}. \tag{73}$$

All required characteristics at the interface can be found by means of the Eqs. (46), (73) by using the formulas (28), (29). Particularly for $x_1 > a$ the following presentation is valid:

$$\begin{aligned}\sigma_{33}^{(1)}(x_1,0) + m_{14}D_3^{(1)}(x_1,0) + im_{11}\sigma_{13}^{(1)}(x_1,0) &= i\theta_1\frac{q_0}{k_0}\left\{\tilde{x}_1 - X(x_1)[d_{11}x_1^2 + d_{12}x_1 + d_{13}]\right\} + i(\theta_3 + m_{14}\theta_4)\frac{q_0}{k_0}\frac{1+\gamma_1}{1-\gamma_1} \\ &\times\left\{\sqrt{(x_1-c)(x_1-a)} - X(x_1)[d_{21}x_1^2 + d_{22}x_1 + d_{23}]\right\} - \frac{q_0}{k_0}\varphi_1(x_1).\end{aligned} \tag{74}$$

From the Eqs. (47b) and (74) the stresses and the electrical displacement at the interface can be found. Introducing the IFs by the following formula:

$$(K_1 + m_{14}K_4) - im_{11}K_2 = (x_1 - a)^{-i\varepsilon}\sqrt{2\pi(x_1 - a)}[\sigma_{33}^{(1)}(x_1,0) + m_{14}D_3^{(1)}(x_1,0) - im_{11}\sigma_{13}^{(1)}(x_1,0)]_{x_1\to a+0} \tag{75}$$

and by using Eq. (74) one arrives at the following expression for the IFs:

$$(K_1 + m_{14}K_4) - im_{11}K_2 = -i(\gamma_1 + 1)\frac{q_0}{k_0}\frac{l\sqrt{\pi l}}{4\sqrt{2}}e^{-i\psi}[(1+2i\varepsilon)^2(\eta_1 + \eta_3) - \eta_3]. \tag{76}$$

From the last equation and Eq. $(55)_2$ (by changing k_1, k_4 for K_1, K_4, respectively) all IFs can be found. Using the expressions (73), (76) and Eq. (29) for $j = 1$ the following asymptotic expression valid for $x_1 \to a - 0$ can be obtained:

$$n_{11}[u_1(x_1,0)] + i\{n_{13}[u_3(x_1,0)] + n_{14}[\varphi(x_1,0)]\} = \frac{2i}{1-2i\varepsilon}\sqrt{\frac{2\alpha}{\pi}}[K_1 + m_{14}K_4 + im_{11}K_2](a - x_1)^{0.5-i\varepsilon}. \tag{77}$$

Combining the last formula with the following expression:

$$n_{43}[u_3(x_1)] + n_{44}[\varphi(x_1)]|_{x_1\to a-0} = \sqrt{\frac{2(a-x_1)}{\pi}}(K_1 + m_{44}K_4) \tag{78}$$

obtained by integration of (58b) and changing k_1, k_4 for K_1, K_4, respectively, the displacement and electrical potential jumps at the crack tip can be found.

For the more general case of a thermoelectromechanical loading the solution obtained by Herrmann et al. (2001) for a pure electromechanical loading should be used in addition to the just reported solution.

8. Numerical results and discussion

The main attention in this paragraph will be devoted to the contact zone model which has never been investigated in the past for an electrically impermeable interface crack in a thermopiezoelectric bimaterial.

For the numerical analysis the same bimaterial as in Part 1, i.e. cadmium selenium/glass has been chosen. For this bimaterial the vector $\boldsymbol{e}$ and the matrix $\boldsymbol{\Omega}$ calculated in the system of physical units SI have the following shape:

$$\boldsymbol{e} = \begin{Bmatrix} -1.38 \times 10^5 \\ -1.88 \times 10^5 \\ -1.85 \times 10^{-7} \end{Bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} -7.43 \times 10^{-11} & -0.0223 \\ -0.0181 & 2.27 \times 10^{10} \end{bmatrix}. \tag{79}$$

Taking into account that e_2 is negative, the sign of q_0 according to the inequalities $(15)_1$ and $(17)_1$ in the following examples will be chosen to be negative. The values of d used in the following examples are controlled by the inequalities $(15)_2$ or $(17)_2$.

In Table 1 the variation of the relative contact zone length λ_1 and the value λ_2, respectively, with respect to the intensity of the temperature flux p for two different values of the shear-normal loading coefficient k and $d = 0$ have been demonstrated, and in Table 2 the normalized SIF of the shear stress

Table 1
The variation of the relative contact zone length λ_1 and the value λ_2, respectively, with respect to p for $d = 0$ and for different shear-normal loading coefficients k for the bimaterial cadmium–selenium/glass

k	0		10	
$-p$	$-\ln(\lambda_1)$	$-\ln(\lambda_2)$	$-\ln(\lambda_1)$	$-\ln(\lambda_2)$
0	69.24	69.27	5.149	5.152
1	68.43	68.45	5.753	5.756
3	61.94	61.97	6.913	6.918
10	52.37	52.41	10.53	10.54
20	48.00	48.04	14.70	14.71
50	44.50	44.54	22.72	22.74
100	43.15	43.18	29.18	29.21
10^3	41.83	41.86	39.97	40.01
10^4	41.69	41.72	41.50	41.53
10^6	41.67	41.70	41.67	41.70

Table 2
The variation of the normalized SIF $(k_{20} + k_{20}^{(em)})/(\sigma\sqrt{l})$ with respect to p for $d = 0$ and for different shear-normal loading coefficients k for the bimaterial cadmium–selenium/glass

$-p$	$k = 0$	$k = 10$
0	1.23	12.6
1	1.43	12.8
3	1.84	13.1
10	3.40	14.4
20	5.71	16.3
50	12.7	22.5
100	24.5	33.6
10^3	2.36×10^2	2.44×10^2
10^4	2.35×10^3	2.36×10^3
10^6	2.35×10^5	2.35×10^5

$(k_{20} + k_{20}^{(\mathrm{em})})/(\sigma\sqrt{l})$ at the right crack tip for $\lambda = \lambda_1$ has been given. It can be clearly seen that the difference between λ_1 and λ_2 is negligible small for all values of p. Moreover, these values like the values of the shear stresses differ very slightly from the associated values obtained in Part 1 for an electrically permeable interface crack.

Further, the influence of the electrical flux d upon the fracture mechanical parameters is demonstrated. Thereby the dimensionless parameter $d^* = c_{33}^{(1)} d/(e_{33}^{(1)}\sigma)$ has been introduced for convenience. In Fig. 1 the variations of the relative contact zone length λ_1 and the value λ_2 with respect to the intensity of the electrical flux d^* for $k = 0$ and two different intensities of the temperature flux are shown. In this graph the logarithmic scale is used because the values of λ_i are extremely small for the considered parameters k and p. It can be seen that an increase of d leads to an increase of the differences between λ_1 and λ_2 which for the left points of each pair of lines are equal to zero.

In Fig. 2 the variations of the normalized intensity factor of the electrical displacement $(k_4 + k_4^{(\mathrm{em})})/(e_{33}^{(1)}\sqrt{l})$ at the point $a = a_1$ $(\lambda = \lambda_1)$ for different intensities of the temperature flux are presented.

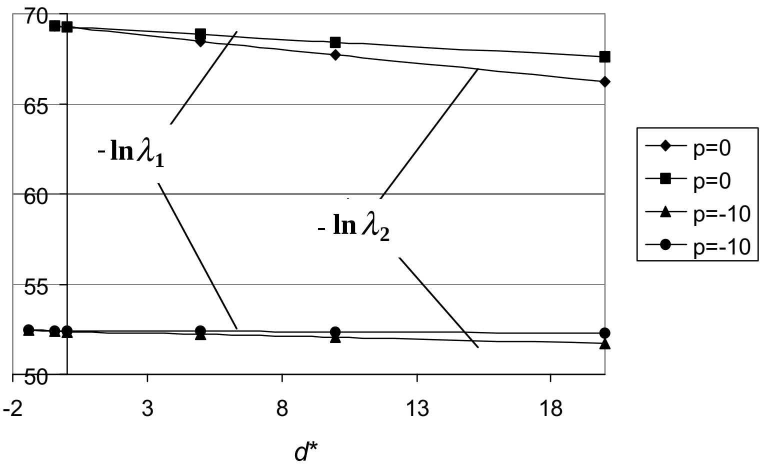

Fig. 1. The variation of the values of λ_i in the logarithmic scale with respect to the intensity of the electrical flux d^*.

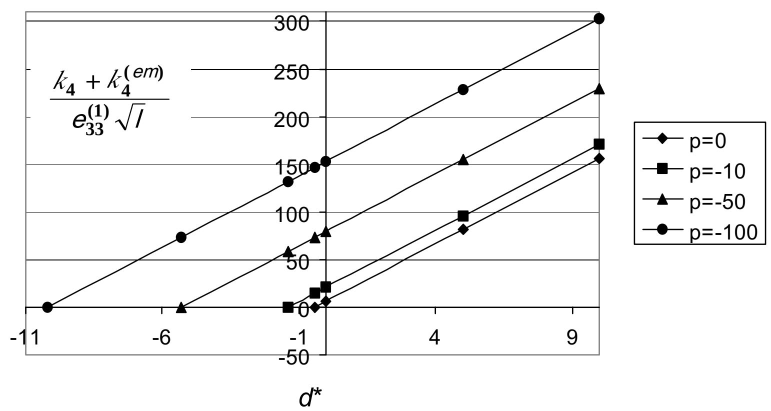

Fig. 2. Variation of the electrical intensity factor at the point a with respect to the intensity of the electrical flux d^* for different intensities of the heat flux.

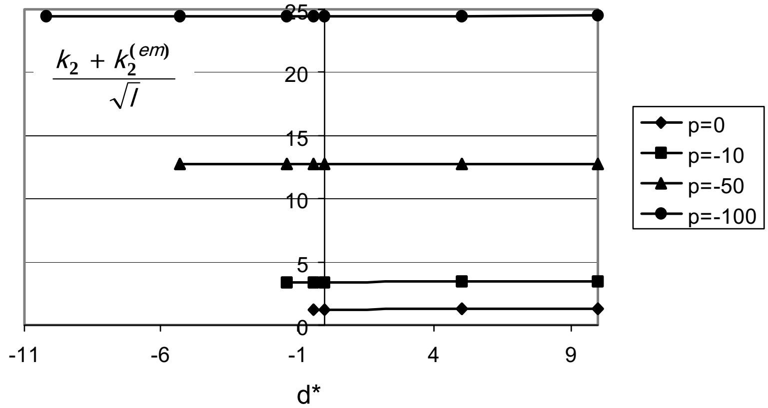

Fig. 3. Variation of the SIF of the shear stress with respect to the intensity of the electrical flux d^* for different intensities of the heat flux.

The values of d^* designated as d_0^* for which $k_4 + k_4^{(\mathrm{em})} = 0$ correspond to the cases when $\lambda_1 = \lambda_2$ ($\Omega_a = a_1 = a_2$) and the interface crack closes smoothly at the point $x_1 = a_1$. These values d_0^* are equal to -0.423, -1.40, -5.31 and -10.19 for p equal to 0, 10, 50 and 100, respectively. A decrease of the d^* value lower then d_0^* leads to negative values of the IF $(k_4 + k_4^{(\mathrm{em})})$ and as a consequence to a violation of the second inequality from (65). In such a case the set Ω_a defined by formula (72) becomes empty and the contact zone model associated with the interface conditions (24) does not exist. This situation is similar to the well-known result concerning the possibility of a transition from a perfect thermal contact of two isotropic bodies to their separation reported for example by Barber and Comninou (1983) and in such a case a new thermoelectrical interface condition should be presented. However, it is worth to mention that for materials considered in this paper and according to Herrmann et al. (2001) the value d_0^* is usually less than zero, and therefore for the most important case, $d = 0$, the interface conditions (24) are applicable.

In Fig. 3 the variations of the normalized SIF $(k_2 + k_2^{(\mathrm{em})})/(\sigma\sqrt{l})$ for $\lambda = \lambda_1$ with respect to the intensity of the electrical flux d^* for $k = 0$ and the same values of p as in Fig. 2 are presented. It is obvious that the analysed values essentially depend on p only, but their dependence on d is rather insensible. It should be mentioned that an essential dependence of the λ_1-values and $(k_2 + k_2^{(\mathrm{em})})$-values on the electrical flux d appears for very large d^*-values only. For example for $p = 0$, $k = 0$ and $d^* = 10^3$, 10^4, 2×10^4 the values of λ_1 are equal to 6.38×10^{-13}, 6.62×10^{-4}, 0.0177, respectively, and $(k_2 + k_2^{(\mathrm{em})})$ varies essentially as well. A further increasing of the d^*-values leads to an increase of λ_1 too.

9. Conclusion

An electrically impermeable interface crack with a contact zone in an infinite piezoelectrical bimaterial under the action of a mixed-mode mechanical loading and a thermoelectrical flux has been considered. Using the matrix-vector representations (26), (27) derived in the first part of this paper the expressions (28), (29) for the required physical values via special sectionally holomorphic functions are found. An auxiliary problem concerning a possible transition from a perfect thermoelectrical contact of two piezoelectric bodies to their thermal and electrical separation has been considered, and the inequalities (15) defining the admissible values of heat and electrical fluxes have been derived.

The main problem of this part of the paper has been reduced to appropriate inhomogeneous combined Dirichlet–Riemann and Hilbert boundary value problems which have been solved exactly in the form

(42)–(45) and (46), respectively. The expressions for the stresses, the electrical displacements and the derivatives of the mechanical displacement and electrical potential jumps at the interface for any contact zone length have been found. Furthermore, the stress and the electrical displacement intensity factors have been determined in an analytical form which becomes especially simple for small values of the contact zone length.

The transcendental equations for the determination of the real contact zone length were obtained and solved for small values of the contact zone length in the form (71). The possibility of a definition of a unique contact zone length was discussed and the conclusion following from a theorem of the minimum of the potential energy has been presented by the root of Eq. (67).

The analytical formulas for the main electromechanical characteristics correspondent to the classical (“open” crack) model were given as well.

The numerical results are presented for the same bimaterial cadmium selenium/glass as in Part I, and provided the absence of an electrical loading they are rather similar to the associated results for an electrically permeable crack. The influence of the electrical flux upon the contact zone length and the associated stress and electrical intensity factors has been demonstrated for the considered bimaterial as well. It was particularly found out that at some extent the electrical flux essentially influences the electrical intensity factor only and its influence upon the contact zone length and the associated stress intensity factor of the shear stresses becomes sensible for rather large values of this quantity only.

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